# Constrained extrema

We look for max and min of a function

$$f : \mathbb{R}^N \longrightarrow \mathbb{R}.$$

in particular $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) \quad \text{or} \quad f(x_1, \dots x_N)$$

constrained to $g(x,y) = 0$

$$f_{|\{g(x,y)=0\}}$$

Graphically, highest and lowest of the surface

$$z = f(x,y)$$

on the equation $g(x,y) = 0$

We might use two methods:

a) Intersection with vertical planes

b) Intersection with horizontal planes (level curves)

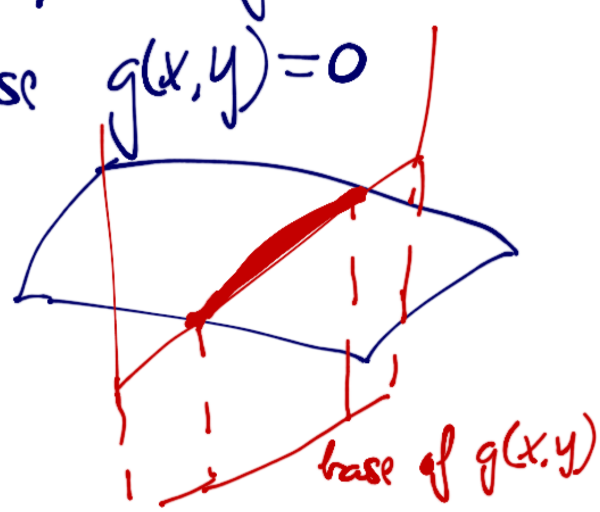
Analitically, this equivalent to

a) Reduce the problem to one variable.

b) Lagrange Multipliers method.

Vertical planes:

Surface $z = f(x,y)$ we cut it by a plane or a cylinder with base $g(x,y)=0$ and find the extrema.



Example: $f(x,y) = x^2 + y^2$

Find the minimum of $f$ constrained to

$$g(x,y) = x + y - 1 = 0$$

Min of $f\big|_{g(x,y)=0}$

We reduce the problem to one variable

and $g(x,y) = \underbrace{x + y - 1 = 0}_{\text{line in } \mathbb{R}^2} \Big| \Rightarrow y = 1 - x$

Substituting into $f(x,y)$

$$f(x,y)\big|_g = x^2 + y^2 = x^2 + (1-x)^2 = x^2 + 1 + x^2 - 2x$$
$$= \boxed{2x^2 - 2x + 1 = h(x)}$$

$$\left.\begin{aligned} h'(x) &= 4x - 2 = 0 \Rightarrow x = \frac{1}{2} \\ y &= 1 - x \Rightarrow y = \frac{1}{2} \end{aligned}\right\}$$

$h''(x) = 4 > 0 \Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right)$ minimum for $f(x,y)$ constrained to $g(x,y)=0$

and the value is

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

**Remark**

We might have problems finding an explicit formula for $g(x,y)=0$

# Intersection with level curves

Looking for maximum and minimum using level curves

$$f(x,y)=c \,,\quad c\in\mathbb{R}.$$

we will move such level curves along the z-axis.

Since we are constrained to

$$g(x,y)=0$$

we must find the intersection of $g(x,y)$ with those level curves:

$$\begin{cases} \bullet \text{ First touching point} \equiv \text{minimum} \\ \bullet \text{ Last touching point} \equiv \text{maximum} \end{cases}$$

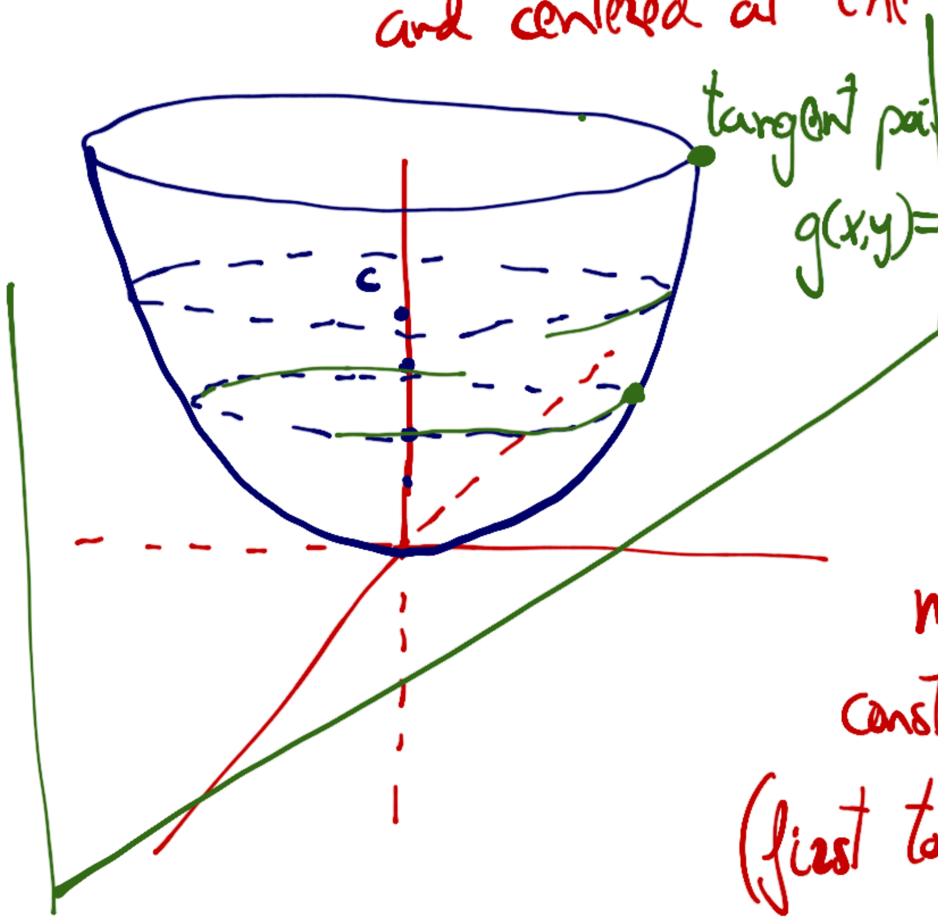
Example: $f(x,y) = x^2+y^2$

$g(x,y) = x+y-1 = 0$

$\left.\begin{array}{l} \\ \end{array}\right\}$ Max and min.

Level curves of $f(x,y)$

$$f(x,y) = \underbrace{x^2+y^2 = c}_{\text{circumferences of radius } \sqrt{c} \text{ and centered at the origin}}$$




$\Downarrow$

min point of $f$ constrained to $g$.
(first touching point)

Lagrange Multiplier method is base on level curves.

Indeed, using level curves we are looking for the points $(x,y)$ where the level curves of the surface

$$z = f(x,y)$$

are tangent to the constrained

$$g(x,y) = 0$$

We know that two curves are tangent if the orthogonal vectors are parallel.

We need to find solutions for $\nabla f(x,y) = \lambda \nabla g(x,y)$

$\lambda \in \mathbb{R}$, $\lambda \equiv$ Lagrange multiplier.

Theorem - Lagrange Multiplier

Let $A \subset \mathbb{R}^n$ be open set in $\mathbb{R}^n$ and

$$f, g : A \subset \mathbb{R}^n \to \mathbb{R}$$

two scalar functions of class $C^1$ in $A$

Let $x_0 \in A$ and $g(x_0) = c$

$S \equiv$ level curve of $g$ at the value $c$

$$S = \{ x \in \mathbb{R}^n ; \ g(x) = c \}$$

If we consider $f$ constrained to $S$

$$f|_S$$

and there is a maximum or minimum $x_0$ for $f|_S$.

Then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda \nabla g(x_0)$$

and $x_0$ is a critical point.

Remark

Lagrangian $\equiv$ $L(x, y, \lambda) = f(x,y) - \lambda g(x,y)$

look for local extrema there.

Example: $f(x,y) = x^2 + y^2$

$\{ g(x,y) = x + y - 1 = 0 \} = S$

$f|_S$ max and min?

Applying Lagrange Multiplier, we must solve the following system:

$$\left.\begin{array}{c} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{array}\right\}$$

$$\Downarrow$$

$$\underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}_{\nabla f(x,y)} = \lambda \underbrace{\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)}_{\nabla g(x,y)}$$

$$\left.\begin{array}{c} (2x, 2y) = \lambda (1, 1) \\ x+y-1 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{c} 2x = \lambda \\ 2y = \lambda \\ x+y-1 = 0 \end{array}\right\}$$

$$\left.\begin{array}{l} 2x = \lambda \\ 2y = \lambda \end{array}\right\} \Rightarrow x = y \quad \text{then} \quad x+y-1=0$$

$$\Downarrow$$

$$2x-1=0$$

$$x = \frac{1}{2} = y.$$

We arrive at only one critical point

$$f\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2} \quad \text{minimum.}$$

## Global extrema

We find extrema in:

- The whole space $\mathbb{R}^n$
- $f\big|_{\{g(x,y)=0\}}$

To guarantee the existence of extrema we need the following result

## Theorem of Weierstrass

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous scalar function in $K \subset \mathbb{R}^N$, $K$ compact set.

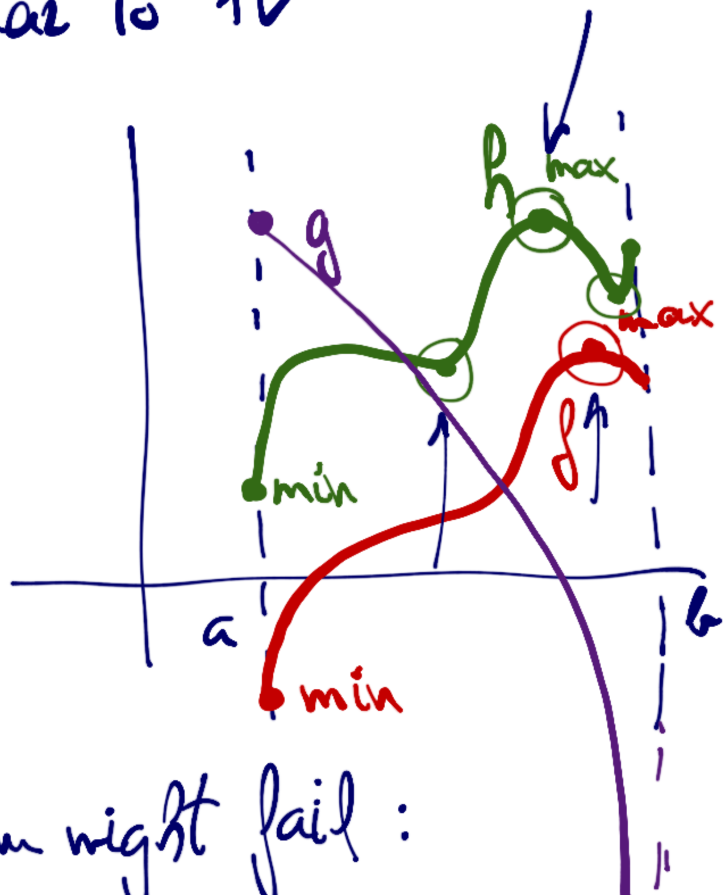
Then there exist $x_m$ and $x_M$ in $K$ such that for any $x \in K$

$$\underbrace{f(x_m) \le f(x) \le f(x_M)}_{f \text{ is bounded.}}$$

and it reaches a maximum and minimum values.

## Remark

Similar to 1D




$[a,b] = K$
compact set

g has a max in $[a,b]$ but not a min because g is not cont. in $[a,b]$.

Theorem might fail:

- If the function is not cont.
- If K is not compact.

$\Downarrow$

we cannot guarantee the existence of max and min.

## Process to find extrema

a) Find critical points in the interior of $A$

$$\nabla f(x_1, \dots x_N) = 0$$

b) Find critical points on the boundary

$f|_{\partial A} \Rightarrow$ Lagrange multiplier/vertical planes.

c) Points where the function is not diff.

min (global)

d) Compare the function $f$ at the points obtained (Weierstrass Theorem)

Example: Find the extrema of the function

$$f(x,y,z) = x+y+z$$

in the ellipsoid $\underbrace{x^2+2y^2+3z^2}_{g(x,y,z)} \leq 1$

First, we observe that

- $f$ is continuous

  and

- $V = \{(x,y,z) \in \mathbb{R}^3,\ x^2+2y^2+3z^2 \leq 1\}$

  is a compact set.

Thanks to Weierstrass Th. $f$ will have a maximum and minimum in $V$

In the interior:

$$\nabla f(x,y,z) = (1,1,1) \neq (0,0,0)$$

There is no critical point in the interior.

So that, there are no extrema in the interior.

On the boundary:

We apply Lagrange Multiplier.

$$\left.\begin{aligned} \nabla f(x,y,z) &= \lambda g(x,y,z) \\ g(x,y,z) &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} (1,1,1) &= \lambda (2x, 4y, 6z) \\ x^2+2y^2+3z^2 &= 1 \end{aligned}\right\}$$

Non-linear system

$$\left|\begin{array}{l} 1 = 2\lambda x \\ 1 = 4\lambda y \\ 1 = 6\lambda z \\ x^2 + 2y^2 + 3z^2 = 1 \end{array}\right\}$$

Obviously $\lambda \neq 0$, so that

$$x = \frac{1}{2\lambda}, \quad y = \frac{1}{4\lambda}, \quad z = \frac{1}{6\lambda}$$

Substituting into the last equation.

$$\frac{1}{4\lambda^2} + \frac{2}{16\lambda^2} + \frac{3}{36\lambda^2} = 1$$

$$\frac{11}{24} = \frac{1}{4} + \frac{1}{8} + \frac{1}{12} = \lambda^2 \Rightarrow \lambda = \pm\sqrt{\frac{11}{24}}$$

We find two points:

$$M_{\pm} = \left( \pm \frac{1}{2}\sqrt{\frac{24}{11}} , \pm \frac{1}{4}\sqrt{\frac{24}{11}} , \pm \frac{1}{6}\sqrt{\frac{24}{11}} \right)$$

$$f(M_{\pm}) = \underline{\pm \frac{11}{12}\sqrt{\frac{24}{11}}} \begin{cases} M_+ & \text{max} \\ M_- & \text{min.} \end{cases}$$

$f$ is also differentiable so that

$$\boxed{\begin{matrix} M_+ \text{ global max} \\ M_- \text{ global min} \end{matrix} \quad \text{for } f|_V}$$

$$f(M_+) = \frac{1}{2}\sqrt{\frac{24}{11}} + \frac{1}{4}\sqrt{\frac{24}{11}} + \frac{1}{6}\sqrt{\frac{24}{11}}$$

$$= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6}\right)\sqrt{\frac{24}{11}} = \frac{1}{12}\overbrace{(6+3+2)}^{11}\sqrt{\frac{24}{11}}$$